# n-sided Surfaces: a Survey

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**Abstract.** The paper surveys techniques for filling in $n$–sided regions, where $n > 4$. The two major classes of methods examined are: 1) to fill in the hole with 4 and/or 3 sided patches, 2) to create a single surface. The multi–patch approaches differ in terms of the degree of the patches and the cross–patch continuity. The single surface approaches are either rational surfaces (which can be expressed in terms of base points) or non-rational, both cases having a number of variants.

## §1. Introduction

The problem being considered is:

*Given $n$ curves $C_1, \dots, C_n$ whose endpoints match, i.e (if we say $C_0 = C_n$) the end of $C_{i-1}$ is the start of $C_i$, fill in the hole bounded by the $C_i$, possibly satisfying some additional boundary conditions.*

For example, in blending, the $C_i$ are the edges of faces, and the filling surface or surfaces must be smooth across the edges.

I will be looking at the case $n > 4$. The problem with no boundary conditions arises in the *cover* command in the ACIS [1] software libraries. ACIS also supports vertex blends using Charrot [6].

This paper extends the survey Malraison [38]. Other surveys include: Nasri [40], Cavaretta [4], Sederberg [57] and Dyn [9] for general overviews of subdivision, Dyn [10] for a review of John Gregory's contributions to the field, Varady [65] is a review of $n$-sided patches, Gregory [13] and Gregory [16] are surveys on $n$-sided patches by Gregory and others, Varady [60] specific to vertex blends and Vida [66] discusses blends in general with a section on $n$-sided issues.

## §2. Subdivision

Subdivision is a much broader topic than I can cover here. The basic process is to generate a surface by starting with a polygonal approximation $P_0$ and having a process which from $P_i$ creates $P_{i+1}$. The $n$-sided problem arises

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in this context when the individual polygons formed by the subdivision process are not 4-sided. Hermann [21] and Ball [2] use subdivision to explicitly fill in $n$-sided holes, while Wang [69,68] uses a single patch method of Varady [65,64] to fill in an $n$-sided hole arising in the course of subdivision. Levin [31,32] applies a *combined* subdivision scheme to solve the $n$-sided problem with cross-tangency constraints. Nasri [41–44] uses subdivision to provide both a source of and solution for $n$-sided problems.

## §3. Multiple Patches

One solution to the problem is to take the $n$-sided region and subdivide it once into triangular or 4-sided regions, and then fill those with standard surface types. The main difficulty with this approach is ensuring the internal smoothness of the resulting network of patches. Peters [45] discusses the problems for doing a $C^K$ join, Varardy [62] looks at curvature matching, and Hall [20,19] looks at the situation when the pieces are Gregory patches. Some of the other approaches are summarized in Table 1.

| Boundary | Degree | Continuity | Reference |
|---|---|---|---|
| Mesh points | Bicubic | $C^1$ | [47] |
| Planes | Biquintic | $C^1$ | [72] |
| Polygon | Cubic | $C^1$ | [8,3] |
| Quartic | Bicubic | $C^1$ | [48–50] |
| Cubic | Quartic | $G^1$ | [52] |
| Cubic | Quartic triangular | $C^1$ | [36,37] |
| Quintic | Quintic | $GC^2$ | [74] |
| Quintic | $2k^2+3k+1$ | $GC^k$ | [76] |
| Quintic | Biquintic | $GC^2$ | [17] |

**Tab. 1.** Multiple patches.

Bangert [3], and Peters [48,49,50] are triangle-based spline methods. Peters [51] adds a hierarchical structure which supports interactive modeling. Some other multi-patch approaches do not fit into the above table. Sone [58,59] subdivides an $n$-sided hole into quadrilaterals and uses Gregory patches. Hsu [24] uses a blend between two edges to fill in a triangular subset of the $n$-sided hole, then continues to subdivide the remaining pieces so that the entire hole is filled in with multiple blend surfaces. This approach is different from the usual multiple patch approach as it may require multiple subdivisions to arrive at regions suitable for blending. Varady [65,64] uses multiple patches for setback blends.

## §4. Single Patches

Filling in an $n$-sided hole with a single patch is an easier approach in a solid modeling environment since only one face need be constructed. The issue here is unusual (*i.e.*, non-rectangular) parametric domains and internal shape control. Hall [20] discusses the control of Gregory patches [13].

For the case where the surface is rational ( *i.e.*, $f(u,v) = \frac{p(u,v)}{q(u,v)}$), Warren [71] shows that several different methods are all variants of rational surfaces with base points ( $p(u,v) = q(u,v) = 0$). Since base points are singularities, they may occur either on the boundary of the domain or outside. S-patches go from a polygon through an n-simplex : $P \xrightarrow{L} \Sigma \xrightarrow{B} R^3$. Those variants are summarized in Table 2.

| **Basepoints** | **Variant** | **Reference** |
|---|---|---|
| Boundary | | |
| | $2n$ | [64] |
| | up to 8 sides | [70] |
| | manifold charts | [12] |
| | 5,6 sides | [54] |
| | includes holes | [29,30] |
| External | | |
| S-Patches: | domain is n-simplex | |
| | original | [35,34] |
| | modifies B | [33] |
| | modifies L | [25,26] |
| Gregory-like: | Polygonal domain | |
| | pentagon | [5] |
| | arbitrary $n$ | [6] |

**Tab. 2.** Single patch.

Gregory [15] starts with a larger problem: interpolating an arbitrary mesh. The solution is to interpolate the "edges" by rational splines to create polygonal curved regions, then extend the splines into strips, and blend the strips into the interior using the same technique as Charrot [5].

The other principal method is to generate a rational surface using a Bezier-like approach by constructing non-rectangular control nets. The boundaries are considered as the edges of Bezier surface patches so higher cross boundary smoothness can be obtained by having the internal control net reflect the adjacent surface control net.

Sabin [53] uses quadratic functions to fill in three-sided and five-sided patches. In Sabin [56] the same technique is applied to a 2-sided patch. Hosaka [23] does the same thing using quadratics and cubics for three-sided, five-sided and six-sided patches. Their solution is described in the general $n$-sided setting. Zheng [73] extends those two approaches by using higher degrees for higher numbers of sides.

Karčiauskas [27,28] and Zube [77,78] provide a unifying approach similar to Warren [71] for these rational cases by looking at toric varieties.

## §5. Other $n$-sided Things

Two papers address a problem which arises in a more global setting from a classical result in topology.

**Theorem.** [39]. *If $M$ is a compact connected 2-manifold, $M$ is a 2-sphere with $h$ handles and $m$ cross caps.*

This theorem implies $M$ can be represented by a polygon with edges identified: e.g. a torus is $ABA^{-1}B^{-1}$. Ferguson [11] applies this result to model objects with a single surface. Wallner [67] applies the same idea to *orbifolds*: surfaces defined as images of group actions. These techniques fall into the scope of this paper insofar as the marked polygon defining the surface is an $n$-sided object in parameter space.

## §6. Conclusions

For single surface patches, Warren [71], Karčiauskas [28], and Zube [78] show that the approaches used so far are variations on two main themes. For multi-patch and subdivision methods no such unifying concept has been presented, although the basic notion of subdivision is arguably the unfiying theme of that approach.

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